

# Segre classes and Kempf-Laksov formula in algebraic cobordism

Thomas Hudson, Tomoo Matsumura

## Abstract

In this paper, we study Segre classes in algebraic cobordism. We also prove a generalization of Kempf-Laksov formula for the degeneracy loci classes in the algebraic cobordism of the Grassmannian bundle.

## 1 Introduction

This work was motivated by the attempt to generalize to algebraic cobordism the Giambelli-Thom-Porteous formula of the Chow ring. Given a vector space of dimension  $n$ , the original Giambelli formula describes the fundamental classes of  $X_\lambda$ , the Schubert varieties of the grassmannian of  $d$ -planes  $Gr(d, n)$ , by means of Schur functions evaluated at the Chern classes of the tautological vector bundle. More specifically, it yields a closed, determinantal expression for  $[X_\lambda]_{CH}$ . This formula was later generalized to the case of the Grassmann bundle  $Gr_d(E)$  associated to a vector bundle  $E$  by Kempf and Laksov in [14]. In this case, given a reference flag  $0 = F^n \subset \dots \subset F^2 \subset F^1 \subset F^0 = E$ , the Schubert classes can be expressed using factorial Schur function in the classes  $c_i(E/F^j - S)$  where  $S$  is the tautological vector bundle of  $Gr_d(E)$ . More explicitly, if  $\lambda = (\lambda_1, \dots, \lambda_r)$  is a partition, we have

$$[X_\lambda]_{CH} = \det (c_{\lambda_i + j - i}(E/F^{\lambda_i - i + d} - S))_{1 \leq i, j \leq r}.$$

We can write the class also by the Vandermonde polynomial and an evaluation map  $\phi_1$ :

$$[X_\lambda]_{CH} = \phi_1 \left( t_1^{\lambda_1} \dots t_r^{\lambda_r} \prod_{1 \leq i < j \leq r} (1 - t_i/t_j) \right)$$

where  $\phi_1$  is a map replacing each monomial  $t_1^{s_1} \dots t_r^{s_r}$  by  $\prod_{i=1}^r c_{s_i}(E/F^{\lambda_i - i + d} - S)$ . The proof essentially consists in the construction, through a tower of projective bundles, of a resolution of singularities  $\tilde{X}_\lambda^{KL} \xrightarrow{\varphi_\lambda} X_\lambda$  and in a Gysin computation which produces the formula for the fundamental class.

These formulas were to become one of the facets of a comprehensive description of the Schubert varieties of flag manifolds  $Fl(\mathbb{C}^n)$  and of flag bundles  $\mathcal{F}\ell(E)$ , respectively established in [1], [5], and [6]. In these cases the fundamental classes are described by the (double) Schubert polynomials of Lascoux-Schützenberger and it should be stressed that in general the expressions are not

closed, but are given via a recursive procedure which exploits a family of varieties  $\tilde{X}_{I(w)}^{BS} \xrightarrow{\psi_{I(w)}} X_w$  known as Bott–Samelson resolutions. At this stage it was noticed that the whole story could be meaningfully rephrased in  $K^0$ , the Grothendieck ring of vector bundles. In fact, in [8] Fulton and Lascoux, using essentially the same proof, managed to express the structure sheaves of the Schubert varieties of  $\mathcal{F}\ell(E)$  using double Grothendieck polynomials.

More recently, after the introduction of the concept of oriented cohomology theories and the construction of algebraic cobordism  $\Omega^*$  by Levine and Morel in [17], several attempts have been made to further generalize these formulas and all associated aspects linked to Schubert calculus. As a matter of fact, both  $CH^*$  and  $K^0[\beta, \beta^{-1}]$  (a graded version of  $K^0$ ) are examples of oriented cohomology theories and several of the techniques used in those classical settings can be translated to cobordism. See [2, 3, 9] for some generalizations related to the flag manifold and [15, 10, 4, 11] for those related to flag bundles.

In these attempts, more caution is required in interpreting the results obtained. On the one hand this is due to the fact that not all Schubert varieties have a well defined notion of fundamental class, only those that are l.c.i. schemes. On the other hand all these techniques inherently depend on the choice of the resolution  $\tilde{X}_{I(w)}^{BS} \xrightarrow{\psi_{I(w)}} X_w$ , which in general is not unique. This is reflected at the polynomial level, since the good stability properties enjoyed by double Schubert and Grothendieck polynomials so far could not be reproduced for  $\Omega^*$ .

To this day essentially all efforts have been devoted to the study of flag bundles, while Grassmann bundles have not received much attention and, more specifically, no one has tried to adapt the computation by Kempf and Laksov. In [12], together with T. Ikeda and H. Naruse, we generalized such computation to connective  $K$ -theory, an oriented cohomology theory obtained from  $\Omega^*$  which can be specialized to both  $CH^*$  and  $K^0$ . In this context we managed to obtain a determinantal formula for the Schubert classes  $[X_\lambda]_{CK}$ . This was achieved by combining the geometric input given by Kempf–Laksov’s resolution  $\tilde{X}_\lambda^{KL} \xrightarrow{\varphi_\lambda} X_\lambda$ , with an algorithmic procedure modelled after the one used by Kazarian in [13] to express the fundamental classes of the Schubert varieties of Grassmann bundles of symplectic and orthogonal type. Given this state of things it looked reasonable to try and see whether or not this procedure could be further generalized to  $\Omega^*$ .

In order to be able to make use of Kazarian’s machinery, we had to introduce a notion of Segre classes for oriented cohomology theories and find a way to describe them as explicitly as possible. The definition we use is the exact analogue of that given by Fulton in [7]. For an oriented cohomology theory  $A^*$  and a bundle  $E \rightarrow X$  of rank  $n$  we set

$$\mathcal{S}_i^A(E) := \pi_* \left( c_1^{i+n-1}(\mathcal{O}(1)) \right),$$

where  $\mathbb{P}^*(E) \xrightarrow{\pi} X$  is the associated projective bundle with tautological quotient bundle  $\mathcal{O}(1)$ . Using Quillen’s formula, which in  $\Omega^*$  was established by Vishik in [19], we manage to relate the Segre polynomial  $\mathcal{S}^A(E; t)$  to the Chern polynomial  $c^A(E; t)$  and the algebraic cobordism classes

of projective spaces. Namely, we have

$$\mathcal{S}^A(E; t) = c^A(-E; -t) \cdot \mathcal{P}^A(t^{-1}) \cdot w^A(-E; t^{-1}) \quad (1.1)$$

with  $\mathcal{P}^A(t) := \sum_i [\mathbb{P}^i]_A \cdot t^{-i}$  and  $w^A(E, t)$  a power series defined at Section 2.3. It is worth noticing that this formula generalizes the classical Chow ring identity  $s_t^{CH}(E) = c_{-t}^{CH}(-E)$ , which allows one to interpret Segre classes as complete symmetric functions in the Chern roots. Needless to say, in general the situation is far more complicated.

An important consequence of this formula is that it allows us to lift the definition of Segre classes to the Grothendieck ring of vector bundles, so that it is possible to evaluate them on virtual bundles. The following result provides a geometric interpretation to this extension.

**Theorem A** [Theorem 3.10]. *Let  $E$  and  $F$  be two vector bundles over  $X$ , respectively of rank  $e$  and  $f$ . Consider the projective bundle  $\mathbb{P}^*(E) \xrightarrow{\pi} X$  with tautological bundle  $\mathcal{O}(1)$ . Then, for every oriented cohomology theory  $A^*$  one has*

$$\pi_* \left( c_f(\mathcal{O}(1) \otimes F^\vee) \right) = \mathcal{S}_{f-e+1}^A(E - F) \quad (1.2)$$

as elements of  $A^*(X)$ .

We now explain our main result. Let  $E$  be a vector bundle of rank  $n$  over a smooth quasi-projective variety  $X$ . Consider the Grassmannian bundle of rank  $d$  subbundles  $\text{Gr}_d(E) \rightarrow X$  and let  $S$  be its the tautological vector bundle. Fix a complete flag  $0 = F^n \subset \dots \subset F^1 \subset E$  where the superscript indicates the corank. Let  $\lambda = (\lambda_1, \dots, \lambda_r)$  be a partition of length  $r$  such that  $\lambda_1 \leq n - d$ . The associated degeneracy locus  $X_\lambda$  in  $\text{Gr}_d(E)$  is defined by

$$X_\lambda := \{(x, S_x) \in \text{Gr}_d(E) \mid \dim(F_x^{\lambda_i - i + d} \cap S_x) \geq i, i = 1, \dots, d\}.$$

Let  $\tilde{X}_\lambda^{KL} \rightarrow X_\lambda \subset \text{Gr}_d(E)$  be the Kempf-Laksov resolution (see Section 4.2). Our explicit closed formula of the cobordism class  $\kappa_\lambda := [\tilde{X}_\lambda^{KL} \rightarrow \text{Gr}_d(E)]$  is as follows. Let  $\mathcal{A}_k^{(\ell)} := \mathcal{S}_k(S^\vee - (E/F^\ell)^\vee)$  for  $k \in \mathbb{Z}$  and  $\ell = 1, \dots, n$ . Let  $P(z, x)$  be a power series in  $z$  and  $x$  such that  $F(z, \chi(x)) = (z - x)P(z, x)$  (see Lemma 2.3).

**Theorem B** [Theorem 4.7]. *We have*

$$[\tilde{X}_\lambda^{KL} \rightarrow \text{Gr}_d(E)] = \phi_1 \left( t_1^{\lambda_1} \dots t_r^{\lambda_r} \prod_{1 \leq i < j \leq r} (1 - t_i/t_j) \prod_{1 \leq i < j \leq r} P(t_j, t_i) \right).$$

where  $\phi_1$  replaces each monomial  $t_1^{s_1} \dots t_r^{s_d}$  by  $\mathcal{A}_{s_1}^{(\lambda_1 - 1 + d)} \dots \mathcal{A}_{s_r}^{(\lambda_r - r + d)}$

Since  $X_\lambda$  has at worst rational singularities, in  $CK^*(\text{Gr}_d(E))$ , the class  $[\tilde{X}_\lambda^{KL} \rightarrow \text{Gr}_d(E)]$  coincides with the class of the degeneracy loci  $X_\lambda$  in  $\text{Gr}_d(E)$ . As mentioned above, generally there is not a well defined notion of the fundamental class of  $X_\lambda$  in the algebraic cobordism. The class  $[\tilde{X}_\lambda^{KL} \rightarrow \text{Gr}_d(E)]$  is going to be our replacement.

The advantage of considering this class is that it is stable along the natural inclusions of the Grassmann bundle into the one for the vector bundles with larger ranks. This allows us to define a generalization of Schur/Grothendieck polynomials in the context of the algebraic cobordism. This aspect of the theory will be studied elsewhere.

## 2 Preliminary on algebraic cobordism

The main goal of this section is to give a brief introduction to algebraic cobordism and its properties. For this purpose we will recall some basic facts about oriented cohomology theories and formal group laws.

### 2.1 Oriented cohomology theories

In algebraic geometry the notion of oriented cohomology theory is due to Levine and Morel, who introduced it in [17], inspired by the work of Quillen on differentiable manifolds ([18]). Such a theory consists of a contravariant functor from the category of smooth schemes to graded abelian rings  $A^* : \mathbf{Sm}_k^{op} \rightarrow \mathcal{R}^*$ , together with a family of push-forward maps  $\{f_* : A^*(X) \rightarrow A^*(Y)\}$  associated to the projective morphisms  $\{X \xrightarrow{f} Y\}$ . This family is supposed to satisfy some straightforward functorial properties and to be compatible with the pull-back maps  $g^*$  arising from  $A^*$ , whenever  $f$  and  $g$  are transverse. Finally,  $A^*$  is supposed to satisfy the extended homotopy property and the projective bundle formula, which respectively relate the evaluation of  $A^*$  on vector and projective bundles to the evaluation on their bases. For the precise definition we refer the reader to [17, Definition 1.1.2].

Fundamental examples of oriented cohomology theories are the Chow ring  $CH^*$  and  $K^0[\beta, \beta^{-1}]$ , a graded version of the Grothendieck ring of vector bundles  $K^0$  obtained by tensoring with the ring of Laurent polynomials  $\mathbb{Z}[\beta, \beta^{-1}]$ , with  $\deg \beta = -1$ . Many of the general features of oriented cohomology theories can be seen already in these two examples. For instance, it is possible to define a theory of Chern classes  $c_i^A$ , which shares many of the properties of the one given in the Chow ring.

In order to understand in which ways the concept of oriented cohomology theory is more general, it can be useful to pay a closer look at the behaviour of the first Chern class of line bundles. It is well known that  $c_1^{CH}$  behaves linearly with respect to tensor product: for line bundles  $L$  and  $M$  over  $X \in \mathbf{Sm}_k$  one has

$$c_1^{CH}(L \otimes M) = c_1^{CH}(L) + c_1^{CH}(M).$$

Although this equality does not necessarily hold if we replace the Chow ring with another oriented cohomology theory  $A^*$ , it is still possible to express  $c_1^A(L \otimes M)$  in terms of the first Chern classes of the factors. However, for this one has to replace the usual sum with a formal group law  $F_A$ .

Before we continue with our discussion, let us now briefly recall the definition of a formal group law. A pair  $(R, F_R)$ , where  $R$  is a ring and  $F_R(u, v) \in R[[u, v]]$ , is said to be a commutative

formal group law of rank 1 if it satisfies the following conditions:

$$i) F_R(u, 0) = F_R(0, u) = u \in R[[u]]; \quad (2.1)$$

$$ii) F_R(u, v) = F_R(v, u) \in R[[u, v]]; \quad (2.2)$$

$$iii) F_R(u, F_R(v, w)) = F_R(F_R(u, v), w) \in R[[u, v, w]]. \quad (2.3)$$

It should be noticed that *ii)* and *iii)* can be viewed as analogues of the commutative and associative properties for groups. Actually, this analogy can be pushed further since there exists also a notion of formal inverse, a power series  $\chi(u) \in R[[u]]$  such that

$$F_R(u, \chi(u)) = 0. \quad (2.4)$$

Let us finish this digression by mentioning that there exists a universal formal group law  $F$ , defined over the Lazard ring  $\mathbb{L}$ , from which all other ones can be derived.

As we hinted at before, every cohomology theory  $A^*$  has an associated formal group law  $F_A$ , defined over its coefficient ring  $A^*(\text{Spec } k)$ , such that in  $A^1(X)$  one has

$$c_1^A(L \otimes M) = F_A(c_1^A(L), c_1^A(M))$$

for any choice of line bundles  $L$  and  $M$  over the given scheme  $X$ . As we have seen  $F_{CH}(u, v) = u + v$ , while for  $K^0[\beta, \beta^{-1}]$  one has  $F_{K^0[\beta, \beta^{-1}]}(u, v) = u + v - \beta \cdot uv$ .

The main achievement of Levine and Morel concerning oriented cohomology theories is the construction of algebraic cobordism, denoted  $\Omega^*$ , which they identify as universal in the following sense.

**Theorem 2.1** ([17, Theorems 1.2.6 and 1.2.7]). *Let  $k$  be a field of characteristic 0.  $\Omega^*$  is universal among oriented cohomology theories on  $\mathbf{Sm}_k$ . That is, for any other oriented cohomology theory  $A^*$  there exists a unique morphism*

$$\vartheta_A : \Omega^* \rightarrow A^*$$

*of oriented cohomology theories. Furthermore, its associated formal group law  $(\Omega^*(\text{Spec } k), F_\Omega)$  is isomorphic to  $(\mathbb{L}, F)$ , the universal one.*

One of the consequences of the universality is that it allows to translate formulas which hold in  $\Omega^*$  to every other oriented cohomology theory  $A^*$ , by making use of  $\vartheta_A$ . In particular, if the given formula has a classical version in either  $CH^*$  or  $K^0$ , then one is supposed to recover it. On the other hand, it is not always the case that properties that hold for the Chow ring or the Grothendieck ring will lift to algebraic cobordism.

For example, one basic instance of this phenomenon can be observed if one tries to compute the fundamental class of some closed subscheme  $Z \xrightarrow{i_Z} X$  in a smooth ambient space. While for the Chow ring it is sufficient to consider any resolution of singularities  $\tilde{Z} \xrightarrow{\varphi_{\tilde{Z}}} X$  to recover  $[Z]_{CH}$  as  $\varphi_{\tilde{Z}*}[\tilde{Z}]_{CH}$ , for  $K^0$  one is able to conclude that  $[\mathcal{O}_Z]_{K^0} = \varphi_{\tilde{Z}*}[\mathcal{O}_{\tilde{Z}}]_{K^0}$  only if  $Z$  has at worst rational

singularities. Even this weaker statement proves to be false for algebraic cobordism, since different desingularizations can yield different push-forward classes. This is precisely what happens when one considers the Schubert varieties of the flag variety and Bott-Samelson resolutions.

On top of this lies an even bigger problem. As mentioned in the introduction, in  $\Omega^*$  a scheme  $Z \xrightarrow{\pi_Z} \operatorname{Spec} k$  has a well defined notion of fundamental class only if it is an l.c.i scheme. In fact, since l.c.i. pullbacks are available, one can make use of  $\Omega_*$ , the homological counterpart of algebraic cobordism which is defined for all quasi-projective schemes (*cf.* [16]). Namely we can set  $[Z]_{\Omega_*} := \pi_Z^*(1)$ , where 1 is viewed as an element in the coefficient ring  $\mathbb{L}$ . Then, as an element of  $\Omega^*(X)$ , the fundamental class of  $[Z]_{\Omega_*}$  is given by  $i_{Z*}([Z]_{\Omega_*})$ , which as a cobordism cycle can be rewritten as  $[Z \xrightarrow{i_Z} X]$ . It is worth noting that, since  $id_{X*} = id_{\Omega^*(X)}$ , for  $Z = X$  one recovers the original definition for smooth schemes  $1_X := [X \xrightarrow{id_X} X]$ .

Let us finish this introductory paragraph by warning that we will follow the common practice of writing  $[X]_{\Omega}$  instead of the more precise notation  $[X \xrightarrow{\pi_X} \operatorname{Spec} k]$  when dealing with the elements of the coefficient ring  $\Omega^*(\operatorname{Spec} k)$ . More generally, the subscript  $\Omega$  will from now on be omitted and, unless stated otherwise, all classes are to be thought as cobordism classes. Finally, we will consider the Lazard ring  $\mathbb{L}$  as a graded ring in view of the isomorphism with  $\Omega^*(\operatorname{Spec} k)$ . *For the rest of the paper, we work with the algebraic cobordism  $\Omega^*$  and  $F(u, v) \in \mathbb{L}[[u, v]]$  denotes the universal formal group law.*

## 2.2 Chern classes

Once one has set  $c_1(L) := s^*s_*(1_X)$  for any line bundle  $L \rightarrow X$  with zero section  $s$ , the existence of a theory of Chern classes is a direct consequence of the projective bundle formula. This claims that, for any given bundle  $E \rightarrow X$  of rank  $n$ , one has the following isomorphism of  $\Omega^*(X)$ -modules:

$$\Omega^*(\mathbb{P}^*(E)) \simeq \bigoplus_{i=0}^{n-1} \xi^i \Omega^*(X).$$

Here  $\xi = c_1(O(1))$  and  $O(1)$  stands for the universal quotient line bundle over  $\mathbb{P}^*(E)$ . Up to a sign, Chern classes should be viewed as the coefficients of the expansion of  $\xi^n$  with respect to the basis  $\{1, \xi, \dots, \xi^{n-1}\}$ . More precisely, together with  $c_0(E) = 1$ , one has in  $\Omega^n(X)$  the following defining equality:

$$\xi^n - c_1(E)\xi^{n-1} + \dots + (-1)^n c_n(E) = 0. \quad (2.5)$$

It can be convenient to assemble together the Chern classes in the Chern polynomial  $c(E; u) := \sum_{i=0}^n c_i(E)u^i$ . In fact, one has the following proposition, usually known as the Whitney product formula.

**Proposition 2.2** ([17, Proposition 4.1.15 (3)]). *Let  $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$  be a short exact sequence of vector bundles over  $X \in \mathbf{Sm}_k$ . Then in  $\Omega^*(X)[u]$  one has*

$$c(F; u) = c(E; u) \cdot c(G; u).$$

Whitney formula guarantees that the assignment  $E \mapsto c(E; u)$  is actually well defined at the level of the Grothendieck group  $K^0(X)$ . As a consequence, one can associate a Chern polynomial to any element of  $K^0(X)$ . In particular for  $[E] - [F]$  one obtains

$$c(E - F; u) = \frac{c(E; u)}{c(F; u)} \quad \text{and} \quad c_i(E - F) = \sum_{j=0}^i c_{i-j}(E) \cdot h_j(F), \quad (2.6)$$

where  $h_j(F)$  stands for the  $j$ -th complete symmetric function in the Chern roots of  $F$ .

### 2.3 Definition of $w(E; u)$

For the computation of Segre classes and the Kempf-Laksov classes, we introduce certain classes  $w_{-s}(E)$  and its generating functions  $w(E; u)$ . The definition of them is based on the following observation.

**Lemma 2.3.** *We have*

$$F(z, \chi(x)) = (z - x)P(z, x)$$

where  $P(z, x)$  is a power series in variables  $z$  and  $x$  with coefficients in  $\mathbb{L}$  of degree 0, with the constant term 1.

*Proof.* To obtain the factorization let us write

$$F(z, \chi(x)) = \sum_{j=0}^{\infty} Q_j(z, x), \quad (2.7)$$

where each  $Q_j(z, x)$  is a homogeneous polynomial of total degree  $j$  in  $z$  and  $x$ . It now suffices to observe that the left handside of (2.7) becomes 0 if one sets  $z = x$ . Therefore we conclude that each  $Q_j$  must be divisible by  $(z - x)$ .  $\square$

**Definition 2.4.** Let  $\mathbf{x} = \{x_1, \dots, x_e\}$  be a set of formal variables. Define  $w_{-s}(\mathbf{x}) \in \mathbb{L}[[\mathbf{x}]]$  by

$$\prod_{q=1}^e P(z, x_q) = \sum_{s \geq 0} w_{-s}(\mathbf{x}) z^s.$$

and let  $w(\mathbf{x}; u) := \sum_{s \geq 0} w_{-s}(\mathbf{x}) u^{-s}$ . Since  $w_0(\mathbf{x})$  has constant term 1, it is invertible in  $\mathbb{L}[[\mathbf{x}]]$ . Therefore we can also define  $\tilde{w}_{-s}(\mathbf{x}; u) = \sum_{s \geq 0} \tilde{w}_{-s}(\mathbf{x}) u^{-s}$  by

$$\tilde{w}_{-s}(\mathbf{x}; u) = \frac{1}{w(\mathbf{x}; u)}.$$

If we set  $x_1, \dots, x_e$  to be Chern roots of a vector bundle  $E$ , then write  $w(E; u) = w(\mathbf{x}; u)$  and  $w_{-s}(E) = w_{-s}(\mathbf{x})$ . Similarly we write  $\tilde{w}(E; u) = \tilde{w}(\mathbf{x}; u)$  and  $\tilde{w}_{-s}(E) = \tilde{w}_{-s}(\mathbf{x})$ . Thus we have

$$c_e(L \otimes E^\vee) = \prod_{q=1}^e F(z, \chi(x_q)) = \frac{\sum_{p=0}^e (-1)^p c_p(E) z^{e-p}}{\tilde{w}(E; z^{-1})}. \quad (2.8)$$

### 3 Segre classes and relative Segre classes

In this section, we study the (relative) Segre classes in the algebraic cobordism. Based on the definition of Segre classes (Definition 3.1), we compute its generating function (Theorem 3.6). Then the relative Segre classes are naturally defined by Definition 3.9, and we also write them as pushforward of Chern classes along a projective bundle (Theorem 3.10). Theorem 3.10 is the main ingredient in the computation of the Kempf-Laksov classes in Section 4.

#### 3.1 Segre class

**Definition 3.1.** Let  $E$  be a vector bundle of rank  $e$  over  $X$ . For each  $m \in \mathbb{Z}$ , consider the dual projective bundle  $\pi : \mathbb{P}^*(E \oplus \mathcal{O}^{\oplus n}) \rightarrow X$  for some  $n \geq \max\{0, -m - e + 1\}$ . Define the degree  $m$  Segre class  $\mathcal{S}_m(E)$  of  $E$  by

$$\mathcal{S}_m(E) = \pi_*(\tau^{m+e+n-1}),$$

where  $\tau$  is the first Chern class of the tautological quotient line bundle  $\mathcal{O}(1)$  of  $\mathbb{P}^*(E \oplus \mathcal{O}^{\oplus n})$ . Denote

$$\mathcal{S}(E; u) = \sum_{m \in \mathbb{Z}} \mathcal{S}_m(E) u^m.$$

**Remark 3.2.** It follows from Vishik's formula and the arguments in its proof ([19, Theorem 5.3]) that the class  $\mathcal{S}_m(E)$  is independent of the choice of  $n$ . Indeed, we may suppose  $m \geq -e + 1$  and consider the bundles  $\pi : \mathbb{P}^*(E) \rightarrow X$  and  $\pi' : \mathbb{P}^*(E \oplus \mathcal{O}) \rightarrow X$ . Let  $\tau$  and  $\tau_1$  be the associated tautological classes respectively. Let  $x_1, \dots, x_e$  be the roots of  $E$  and  $x_{e+1}$  be the root of  $\mathcal{O}$ . Since  $x_{e+1} = 0 \in \Omega^*(X)$  and  $F(x_k, \chi(x_{e+1})) = x_k$  for  $k \neq e + 1$ , we have

$$\pi'_*(\tau_1^{m+e}) = \sum_{k=1}^{e+1} \frac{x_k^{m+e}}{\prod_{\substack{1 \leq q \leq e+1 \\ q \neq k}} F(x_k, \chi(x_q))} = \sum_{k=1}^e \frac{x_k^{m+e-1}}{\prod_{\substack{1 \leq q \leq e \\ q \neq k}} F(x_k, \chi(x_q))} = \pi_*(\tau^{m+e-1}).$$

In order to compute  $\mathcal{S}(E; u)$ , we introduce the following series.

**Definition 3.3.** Define the series  $R(E; u) = \sum_{m \in \mathbb{Z}} R_m(E)$  by

$$R(E; u) := \mathcal{S}(E; u) c(E; -u), \quad \text{i.e.} \quad R_m(E) := \sum_{p=0}^e (-1)^p c_p(E) \mathcal{S}_{m-p}(E).$$

**Lemma 3.4.** We have  $R_m(E) = 0$  for  $m > 0$ .

*Proof.* This follows immediately from (2.5) in the projection bundle formula. Indeed,

$$\sum_{p=0}^{\infty} (-1)^p c_p(E) \mathcal{S}_{m-p}(E) = \pi_* \left( \tau^{m-1} \sum_{p=0}^{\infty} (-1)^p c_p(E) \tau^{e+n-p} \right) = \pi_* (\tau^{m-1} \cdot 0) = 0.$$

□



Furthermore, let

$$\mathcal{P}(u) := \sum_{i=0}^{\infty} [\mathbb{P}^i] u^{-i}.$$

where  $[\mathbb{P}^i]$  is the class of the projective space  $\mathbb{P}^i$  of degree  $-i$  in  $\mathbb{L} = \Omega^*(\text{Spec } k)$  for each integer  $i \geq 0$ .

**Proposition 3.5.** *We have  $R(E; u) = \mathcal{P}(u) \tilde{w}(E; u)$ . Or equivalently, we have*

$$R_{-\ell}(E) = \sum_{s \geq 0}^{\ell} \tilde{w}_{-s}(E) [\mathbb{P}^{\ell-s}].$$

*Proof.* Let  $\ell \geq 0$  and let  $F$  be a vector bundle of a smooth quasi-projective space  $Y$  of rank  $\ell + 1$ . Let  $x_1, \dots, x_e$  be the roots of  $E$  and  $y_1, \dots, y_{\ell+1}$  be the roots of  $F$ . We denote  $y_i$  also by  $x_{e+i}$  for  $i = 1, \dots, \ell + 1$ . Consider the dual projective bundle  $\pi : \mathbb{P}^*(E \oplus F) \rightarrow X \times Y$ . Let  $\tau$  be its tautological class. Choose a point  $y \in Y$  and let  $f : X \rightarrow X \times Y$  be the natural inclusion  $f(x) := (x, y)$ . We may assume that  $F(x_i, \chi(x_j))$  is a non-zero divisor in  $\Omega^*(X \times Y)$  for  $i \neq j$ . Then since  $f^*c_p(E \oplus F) = c_p(E)$  and  $f^*\mathcal{S}_m(E \oplus F) = \mathcal{S}_m(E)$  (cf. [19, p.549]), we have

$$R_{-\ell}(E) = \sum_{p=0}^e (-1)^p c_p(E) f^* \left( \sum_{k=1}^{e+\ell+1} \frac{x_k^{e-p}}{\prod_{\substack{q=1, \dots, e+\ell+1 \\ q \neq k}} F(x_k, \chi(x_q))} \right).$$

Since  $\sum_{p=0}^e (-1)^p c_p(E) x_k^{e-p} = 0$  if  $k = 1, \dots, e$ , we have

$$R_{-\ell}(E) = \sum_{p=0}^e (-1)^p c_p(E) f^* \left( \sum_{k=1}^{\ell+1} \frac{y_k^{e-p}}{\prod_{q=1}^e F(y_k, \chi(x_q)) \prod_{\substack{q=1, \dots, \ell+1 \\ q \neq k}} F(y_k, \chi(y_q))} \right).$$

Then, regarding  $E$  as a bundle over  $X \times Y$ , Equation (2.8) allows us to write

$$\frac{\sum_{p=0}^e (-1)^p c_p(E) y_k^{e-p}}{\prod_{q=1}^e F(y_k, \chi(x_q))} = \sum_{s \geq 0}^{\infty} \tilde{w}_{-s}(E) y_k^s.$$

Therefore, we obtain

$$R_{-\ell}(E) = \sum_{s \geq 0}^{\infty} \tilde{w}_{-s}(E) f^* \left( \sum_{k=1}^{\ell+1} \frac{y_k^s}{\prod_{q \neq k} F(y_k, \chi(y_q))} \right).$$

We observe that

$$f^* \left( \sum_{k=1}^{\ell+1} \frac{y_k^s}{\prod_{q \neq k} F(y_k, \chi(y_q))} \right) = \begin{cases} [\mathbb{P}^{\ell-s}] & 0 \leq s \leq \ell, \\ 0 & s > \ell. \end{cases}$$

Thus

$$R_{-\ell}(E) = \sum_{s \geq 0}^{\ell} \tilde{w}_{-s}(E) [\mathbb{P}^{\ell-s}].$$

□

Definition 3.3 and Proposition 3.5 give us the following description of the generating function of the Segre classes.

**Theorem 3.6.** *For a vector bundle  $E$  over  $X$ , in  $\Omega^*(X)$ , the following identity holds:*

$$\mathcal{S}(E; u) = \mathcal{P}(u) \frac{\tilde{w}(E; u)}{c(E; -u)}$$

**Example 3.7.** In the connective  $K$ -theory  $CK^*(X)$ , we have  $\mathcal{P}(u) = \frac{1}{1-\beta u^{-1}}$ ,  $\tilde{w}(E; u) = c(E; \beta)$ . Thus Theorem 3.6 gives

$$\mathcal{S}(E; u) = \frac{1}{1 - \beta u^{-1}} \frac{c(E; -\beta)}{c(E; -u)},$$

which has been obtained in [12]. Note that the sign of  $\beta$  is opposite from the one in [12].

### 3.2 Relative Segre class

Let  $0 \rightarrow E \rightarrow G \rightarrow F \rightarrow 0$  be a short exact sequence of vector bundles. From Definition 2.4, we can observe that

$$w(G; u) = w(E; u)w(F; u), \quad \tilde{w}(G; u) = \tilde{w}(E; u)\tilde{w}(F; u).$$

This allows us to define the following.

**Definition 3.8.** For arbitrary vector bundles  $E$  and  $F$ , define the relative classes  $w_s(E - F)$  and  $\tilde{w}_s(E - F)$  for the  $K$ -class  $[E - F]$  in the Grothendieck group of vector bundles over  $X$  by

$$w(E - F; u) := \sum_{s=0}^{\infty} w_{-s}(E - F) u^{-s} := \frac{w(E; u)}{w(F; u)}, \quad \tilde{w}(E - F; u) := \sum_{s=0}^{\infty} \tilde{w}_{-s}(E - F) u^{-s} := \frac{\tilde{w}(E; u)}{\tilde{w}(F; u)}$$

**Definition 3.9** (Relative Segre classes). Define  $\mathcal{S}_m(E - F)$  by

$$\mathcal{S}(E - F; u) := \sum_{m \in \mathbb{Z}} \mathcal{S}_m(E - F) u^m := \mathcal{S}(E; u) c(F; -u) w(F; u). \quad (3.1)$$

The following description generalizes Proposition 1 in [12] to the algebraic cobordism.

**Theorem 3.10.** *Let  $E$  and  $F$  be vector bundles over  $X$  with rank  $e$  and  $f$  respectively. Let  $\pi : \mathbb{P}^*(E) \rightarrow X$  be the dual projective bundle,  $\mathcal{Q}$  its tautological quotient line bundle, and  $\tau := c_1(\mathcal{Q})$ . We have*

$$\pi_*(\tau^s c_f(\mathcal{Q} \otimes F^\vee)) = \mathcal{S}_{f-e+1+s}(E - F).$$

*Proof.* By Definition 2.4,

$$\tau^s c_e(\mathcal{Q} \otimes F^\vee) = \sum_{q=0}^f \sum_{j=0}^{\infty} (-1)^q c_q(F) w_{-j}(F) \tau^{j+f-q+s}.$$

Thus, by the definition of  $\mathcal{S}_m(E)$ , we have

$$\pi_*(\tau^s c_f(\mathcal{Q} \otimes F^\vee)) = \sum_{q=0}^f \sum_{j=0}^{\infty} (-1)^q c_q(F) w_{-j}(F) \mathcal{S}_{f-e+1+s-q+j}(E),$$

the right hand side of which is  $\mathcal{S}_{f-e+1+s}(E - F)$  by (3.1).  $\square$

## 4 Kempf-Laksov determinant formula

### 4.1 Degeneracy loci

Let  $E$  be a vector bundle of rank  $n$  over a smooth quasi-projective variety  $X$ . Let  $\xi : \text{Gr}_d(E) \rightarrow X$  be the Grassmannian bundle of rank  $d$  subbundles over  $X$ , *i.e.*

$$\text{Gr}_d(E) := \{(x, S_x) \mid x \in X, S_x \text{ is a } d\text{-dimensional subspace of } E_x\}.$$

Let  $S$  be the tautological subbundle of  $\xi^*E$  over  $\text{Gr}_d(E)$ . Fix a complete flag  $0 = F^n \subset \dots \subset F^1 \subset F^0 = E$  where the superscript indicates the corank, *i.e.*  $\text{rk } F^k = n - k$ . We denote also by  $E$  and  $F^i$  the pullback of  $E$  and  $F^i$  along  $\xi$  respectively.

Let  $\mathcal{P}_d$  be the set of all partitions  $(\lambda_1, \dots, \lambda_d)$  with at most  $d$  parts. The length of  $\lambda \in \mathcal{P}_d$  is the number of nonzero parts. Let  $\mathcal{P}_d(n)$  be the set of all partitions in  $\mathcal{P}_d$  such that  $\lambda_i \leq n - d$  for all  $i = 1, \dots, d$ .

For each  $\lambda \in \mathcal{P}_d(n)$  of length  $r$ , consider the partial flag of  $E$

$$F_\lambda^\bullet : F^{\lambda_1-1+d} \subset F^{\lambda_2-2+d} \subset \dots \subset F^{\lambda_r-r+d} \subset E.$$

Define the type  $A$  degeneracy locus  $X_\lambda$  in  $\text{Gr}_d(E)$  by

$$X_\lambda := \{(x, S_x) \in \text{Gr}_d(E) \mid \dim(F_x^{\lambda_i-i+d} \cap S_x) \geq i, i = 1, \dots, r\}.$$

### 4.2 The class of the Kempf-Laksov resolution

Let  $\lambda \in \mathcal{P}_d(n)$  of length  $r$ . Associated to the partial flag  $F_\lambda^\bullet$  is a generalized flag bundle

$$\varpi : \text{Fl}(F_\lambda^\bullet) \rightarrow \text{Gr}_d(E),$$

such that the fiber at  $p \in \text{Gr}_d(E)$  consists of flags of subspaces  $(D_1)_p \subset \dots \subset (D_r)_p$  of  $E_p$  with  $\dim(D_i)_p = i$  and  $(D_i)_p \subset F_p^{\lambda_i-i+d}$ . Let  $D_1 \subset \dots \subset D_r$  be the corresponding flag of tautological subbundles over  $\text{Fl}(F_\lambda^\bullet)$ . One can obtain the flag bundle  $\text{Fl}(F_\lambda^\bullet)$  as a tower of projective bundles

$$\begin{aligned} \varpi : \text{Fl}(F_\lambda^\bullet) &= \mathbb{P}(F^{\lambda_r-r+d}/D_{r-1}) \xrightarrow{\varpi_r} \mathbb{P}(F^{\lambda_{r-1}-(r-1)+d}/D_{r-2}) \xrightarrow{\varpi_{r-1}} \dots \\ &\dots \xrightarrow{\varpi_3} \mathbb{P}(F^{\lambda_2-2+d}/D_1) \xrightarrow{\varpi_2} \mathbb{P}(F^{\lambda_1-1+d}) \xrightarrow{\varpi_1} \text{Gr}_d(E). \end{aligned} \quad (4.1)$$

We regard  $D_i/D_{i-1}$  as the tautological line bundle  $\mathcal{O}(-1)$  of  $\mathbb{P}(F^{\lambda_i-i+d}/D_{i-1})$ .

**Definition 4.1.** Let  $\tilde{X}_\lambda^{KL} \subset \text{Fl}(F_\lambda^\bullet)$  be the locus where  $D_r \subset S$ . It is well-known that  $\tilde{X}_\lambda^{KL}$  is smooth and birational to  $X_\lambda$  along  $\varpi$  (see [14]). We call it the Kempf-Laksov resolution of the degeneracy loci  $X_\lambda$ . Define the Kempf-Laksov class  $\kappa_\lambda \in \Omega^*(\text{Gr}_d(E))$  associated to a partition  $\lambda \in \mathcal{P}_d(n)$  by

$$\kappa_\lambda := [\tilde{X}_\lambda^{KL} \rightarrow \text{Gr}_d(E)] = \varpi_*[\tilde{X}_\lambda^{KL} \rightarrow \text{Fl}(F_\lambda^\bullet)].$$

The next proposition follows from Lemma 4.3 below and the same argument in [12, Section 4.3].

**Proposition 4.2.** *In  $\Omega^*(\mathrm{Fl}(F_\lambda^\bullet))$ , we have*

$$[\tilde{X}_\lambda^{KL} \rightarrow \mathrm{Fl}(F_\lambda^\bullet)] = \prod_{i=1}^r c_{n-d}((D_i/D_{i-1})^\vee \otimes E/S). \quad (4.2)$$

**Lemma 4.3** (Lemma 6.6.7 [17], Example 14.1.1 [7]). *Let  $V$  be a vector bundle of rank  $m$  over  $X$  and  $s$  a section of  $V$ . Let  $Z$  be the zero scheme of  $s$ . If  $X$  is Cohen-Macaulay and the codimension of  $Z$  is  $e$ , then  $s$  is regular and*

$$c_d(V) = [Z] \in \Omega^e(X).$$

**Remark 4.4.** When we specialize the class  $\kappa_\lambda$  in  $CK^*(\mathrm{Gr}_d(E))$ , it agrees with the class  $[X_\lambda]_{CK}$  of the degeneracy loci  $X_\lambda$  (cf. [12]).

We will compute the class  $\kappa_\lambda$  by pushing forward the product of Chern classes (4.2) through the tower of projective bundles (4.1). For that, first we write the following formula for each stage of the tower.

**Lemma 4.5** (The  $i$ -th stage pushforward formula). *Let  $\alpha_i := c_{n-d}((D_i/D_{i-1})^\vee \otimes E/S)$  and  $\tau_i := c_1((D_i/D_{i-1})^\vee)$  in  $\Omega^*(\mathbb{P}(F^{\lambda_i-i+d}/D_{i-1}))$ . Let*

$$\mathcal{A}_k^{(\ell)} := \mathcal{S}_k(S^\vee - (E/F^\ell)^\vee)$$

for  $k \in \mathbb{Z}$  and  $\ell = 1, \dots, n$ . For each non-negative integer  $s$ , we have

$$\varpi_{i*}(\tau_i^s \alpha_i) = \sum_{q=0}^{i-1} \sum_{j=0}^{\infty} (-1)^q c_q(D_{i-1}^\vee) w_{-j}(D_{i-1}^\vee) \mathcal{A}_{\lambda_i+s-q+j}^{(\lambda_i-i+d)}. \quad (4.3)$$

*Proof.* We apply Theorem 3.10 to  $\mathbb{P}^*((F^{\lambda_i-i+d}/D_{i-1})^\vee) = \mathbb{P}(F^{\lambda_i-i+d}/D_{i-1})$  with the tautological quotient bundle  $(D_i/D_{i-1})^\vee$ . We have

$$\begin{aligned} \varpi_{i*}(\tau_i^s \cdot c_{n-d}((D_i/D_{i-1})^\vee \otimes E/S)) &= \mathcal{S}_{\lambda_i+s}((F^{\lambda_i-i+d}/D_{i-1} - E/S)^\vee) \\ &= \mathcal{S}_{\lambda_i+s}((S - E/F^{\lambda_i-i+d})^\vee - D_{i-1}^\vee). \end{aligned}$$

Now the claim follows by the definition of the relative Segre class (3.1).  $\square$

### 4.3 Computing the class $\kappa_\lambda$

In order to systematically apply Lemma 4.5 and obtain a formula for  $\kappa_\lambda$ , we need some preparation of algebras, following [12]. Let  $R = \Omega^*(\mathrm{Gr}_d(E))$ . This is a graded algebra over  $\mathbb{L}$ .

Let  $t_1, \dots, t_r$  be indeterminates of degree 1. We use the multi-index notation  $t^{\mathbf{s}} := t_1^{s_1} \cdots t_r^{s_r}$  for  $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{Z}^r$ . A formal Laurent series

$$f(t_1, \dots, t_r) = \sum_{\mathbf{s} \in \mathbb{Z}^r} a_{\mathbf{s}} t^{\mathbf{s}}$$

is *homogeneous of degree*  $m \in \mathbb{Z}$  if  $a_s$  is zero unless  $a_s \in R_{m-|s|}$  with  $|s| = \sum_{i=1}^r s_i$ . Let  $\text{supp} f = \{s \in \mathbb{Z}^r \mid a_s \neq 0\}$ .

For each  $m \in \mathbb{Z}$ , define  $\mathcal{L}_m^R$  to be the space of all formal Laurent series of homogeneous degree  $m$  such that there exists  $n \in \mathbb{Z}^r$  such that  $n + \text{supp} f$  is contained in the cone in  $\mathbb{Z}^r$  defined by  $s_1 \geq 0, s_1 + s_2 \geq 0, \dots, s_1 + \dots + s_r \geq 0$ . Then  $\mathcal{L}^R := \bigoplus_{m \in \mathbb{Z}} \mathcal{L}_m^R$  is a graded ring over  $R$  with the obvious product.

For each  $i = 1, \dots, r$ , let  $\mathcal{L}^{R,i}$  be the  $R$ -subring of  $\mathcal{L}^R$ , consisting of series that do not contain the negative powers of  $t_1, \dots, t_{i-1}$ . In particular,  $\mathcal{L}^{R,1} = \mathcal{L}^R$ .

A series  $f(t_1, \dots, t_r)$  is a *power series* if it doesn't contain any negative power of  $t_1, \dots, t_r$ . Let  $R[[t_1, \dots, t_r]]_m$  denote the set of all power series in  $t_1, \dots, t_r$  of degree  $m \in \mathbb{Z}$ . We define

$$R[[t_1, \dots, t_r]]_{\text{gr}} := \bigoplus_{m \in \mathbb{Z}} R[[t_1, \dots, t_r]]_m.$$

For each  $i \in \{1, \dots, r\}$ , we define a graded  $R[[t_1, \dots, t_{i-1}]]_{\text{gr}}$ -module homomorphism

$$\phi_i : \mathcal{L}^{R,i} \rightarrow \mathbb{P}(F^{\lambda_{i+1}-i+1+d}/D_{i-2})$$

by  $\phi_i(t_1^{s_1} \dots t_r^{s_d}) = \tau_1^{s_1} \dots \tau_{i-1}^{s_{i-1}} \mathcal{A}_{s_i}^{(\lambda_i-i+d)} \dots \mathcal{A}_{s_r}^{(\lambda_r-r+d)}$ . Then we can write (4.3) using  $\phi_i$  as follows.

**Lemma 4.6.** *We have*

$$\varpi_{i*}(\tau_i^s \alpha_i) = \phi_i \left( t_i^{\lambda_i+s} \prod_{\ell=1}^{i-1} (1 - t_\ell/t_i) P(t_\ell, t_i) \right).$$

*Proof.* By Equation (4.3) and the definition of  $\phi_i$ , we have

$$\begin{aligned} \varpi_{i*}(\tau_i^s \alpha_i) &= \phi_i \left( \sum_{q=0}^{i-1} \sum_{j=0}^{\infty} (-1)^q e_q(t_1, \dots, t_{i-1}) w_{-j}(t_1, \dots, t_{i-1}) t_i^{\lambda_i+s-q+j} \right) \\ &= \phi_i \left( \left( \sum_{q=0}^{i-1} (-1)^q e_q(t_1, \dots, t_{i-1}) t_i^{-q} \right) \left( \sum_{j=0}^{\infty} w_{-j}(t_1, \dots, t_{i-1}) t_i^j \right) t_i^{\lambda_i+s} \right). \end{aligned}$$

The claim follows from the definitions of  $w_{-j}$  and the elementary symmetric polynomials  $e_q$  in terms of the generating functions.  $\square$

Finally, we obtain our main result.

**Theorem 4.7.** *For a partition  $\lambda \in \mathcal{P}_d(n)$ , the associated Kempf-Laksov class  $\kappa_\lambda$  is given by*

$$\kappa_\lambda = \phi_1 \left( t_1^{\lambda_1} \dots t_r^{\lambda_r} \prod_{1 \leq i < j \leq r} (1 - t_i/t_j) \prod_{1 \leq i < j \leq r} P(t_j, t_i) \right).$$

*Proof.* By Definition 4.1 and Proposition 4.2, we have

$$\kappa_\lambda = \varpi_{1*} \circ \dots \circ \varpi_{r*} \left( \prod_{i=1}^r \alpha_i \right).$$

Now the claim follows from the consecutive application of Lemma 4.6 (cf. [12, Section 4.4]).  $\square$

## References

- [1] BERNŠTEĬN, I. N., GEL'FAND, I. M., AND GEL'FAND, S. I. Schubert cells, and the cohomology of the spaces  $G/P$ . *Russian Math. Surveys* 28 (1973), 1–26.
- [2] BRESSLER, P., AND EVENS, S. Schubert Calculus in Complex Cobordism. *Trans. Amer. Math. Soc.* 331, 2 (1992), 799–813.
- [3] CALMÈS, B., PETROV, V., AND ZAINOULLINE, K. Invariants, torsion indices and oriented cohomology of complete flags. *ArXiv e-prints* (May 2009).
- [4] CALMÈS, B., ZAINOULLINE, K., AND ZHONG, C. Equivariant oriented cohomology of flag varieties. *Doc. Math.*, Extra vol.: Alexander S. Merkurjev's sixtieth birthday (2015), 113–144.
- [5] DEMAZURE, M. Désingularisation des variétés de Schubert généralisées. *Ann. Sci. École Norm. Sup. (4)* 7 (1974), 53–88. Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I.
- [6] FULTON, W. Flags, Schubert polynomials, degeneracy loci, and determinantal formulas. *Duke Math. J.* 65, 3 (1992), 381–420.
- [7] FULTON, W. *Intersection theory*, second ed., vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1998.
- [8] FULTON, W., AND LASCOUX, A. A Pieri formula in the Grothendieck ring of a flag bundle. *Duke Math. J.* 76, 3 (1994), 711–729.
- [9] HORNBOSTEL, J., AND KIRITCHENKO, V. Schubert calculus for algebraic cobordism. *J. Reine Angew. Math.* 656 (2011), 59–85.
- [10] HUDSON, T. A Thom-Porteous formula for connective  $K$ -theory using algebraic cobordism. *Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology* 14 (10 2014), 343–369.
- [11] HUDSON, T. Generalised symplectic Schubert classes. *ArXiv e-prints* (Apr. 2015).
- [12] HUDSON, T., IKEDA, T., MATSUMURA, T., AND NARUSE, H. Determinantal and Pfaffian formulas of K-theoretic schubert calculus. 2015, [arXiv:1504.02828v2](#).
- [13] KAZARIAN, M. On lagrange and symmetric degeneracy loci. *Isaac Newton Institute for Mathematical Sciences Preprint Series* (2000).

- [14] KEMPF, G., AND LAKSOV, D. The determinantal formula of Schubert calculus. *Acta Math.* 132 (1974), 153–162.
- [15] KIRITCHENKO, V., AND KRISHNA, A. Equivariant cobordism of flag varieties and of symmetric varieties. *Transform. Groups* 18, 2 (2013), 391–413.
- [16] LEVINE, M. Intersection theory in algebraic cobordism. 2015, [arXiv:1512.08911](#).
- [17] LEVINE, M., AND MOREL, F. *Algebraic cobordism*. Springer Monographs in Mathematics. Springer, Berlin, 2007.
- [18] QUILLEN, D. G. Elementary proofs of some results of cobordism theory using Steenrod operations. *Advances in Math.* 7 (1971), 29–56.
- [19] VISHIK, A. Symmetric operations in algebraic cobordism. *Adv. Math.* 213, 2 (2007), 489–552.